A dispersion relation for waves of finite amplitude in a two-stream plasma

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A dispersion relation is derived for plane space-charge waves of finite amplitude in a plasma containing two oppositely charged streams of particles. The relation may be expressed simply in parametric form; it can also be quite well approximated, in two different ranges of amplitude, by expressions for the wavelength in terms of the maximum variation of the electrostatic potential.

1. Introduction

Much interest has been shown, during the past few years, in the properties of space-charge waves in a two-stream plasma. This is the simplest kind of plasma with a velocity dispersion, and yet it shows up some of the most common properties of plasmas in general. For example, linearized theory predicts that an oscillation in such a plasma can be exponentially amplified when its wavelength is below a certain critical value, determined by the densities and the relative velocity of the two streams, and the masses and the charges of the particles in them (see, for example, Kahn 1957). Buneman (1958) has proposed that this tendency to instability may prohibit too rapid a flow of electrons past the ions in a plasma, and may thus set a limit to the current density which can be attained. However, it has not apparently been possible yet to verify by experiment that the predicted instability exists, because the spread of velocities in each plasma stream cannot yet be made sufficiently small (Nexsen, Cummins, Coensgen & Sherman 1960).

An analytical expression, in parametric form, is derived in this paper for the dispersion relation of a two-stream plasma. The result is valid for finite amplitudes, but applies only to cases in which the two plasma streams are oppositely charged, and where neither set of particles is trapped by the wave. We shall consider only steady waves; however, the formulae here established can probably be used to study some classes of amplified space charge waves to an approximation higher than the linear one, and the author hopes to do this in another paper.

It is, of course, known that the problem of the construction of a plasma wave of finite and constant amplitude can be reduced to that of performing a quadrature (see, for example, Bernstein, Greene & Kruskal 1957). Thus let the wave pattern be at rest in the chosen co-ordinate system, let the mean densities of particles in the streams be $\bar{n}_r$, the particle masses m/μ_r, the particle charges $Z_r e$ and the particle energies χ_r ($r = 1, 2, 3, \ldots$). At a point where the electrostatic potential is V, the particle velocities will be

$$u_r = \sqrt{\{(2\mu_r/m)(\chi_r - Z_r e V)\}}. \tag{1}$$

We restrict our further discussion to the case in which none of the particles are trapped, so that u_r never vanishes or changes sign. If F_r is the particle flux in the rth stream, the particle density there will be

$$n_r = F_r/u_r, \tag{2}$$

and then Poisson's equation gives that

$$\frac{d^2V}{dx^2} = -4\pi e \Sigma \frac{Z_r F_r}{u_r}, \tag{3}$$

provided there is overall space charge neutrality, or

$$\Sigma \bar{n}_r Z_r = 0. \tag{4}$$

The right-hand side of (3) is a function of V only; the first integral of this equation can therefore be found immediately. A further quadrature gives x as a function of V, and, on inversion, V as a function of x. Thus an expression can be found for the wavelength of the plasma wave; the result will then be given in terms of the particle fluxes F_r and energies χ_r. From these results one can also find the values of $\bar{n}_r$ and of the mean velocities $\bar{u}_r$ of the different kinds of particle, and in principle the dispersion relation can then be expressed in terms of them, rather than in terms of F_r and χ_r, if desired.

But the process we have described involves much numerical work. In the next section we shall show how this may be avoided for the case of a two-stream plasma.*

2. A derivation of the dispersion relation

We begin the derivation with the help of the equations of motion and continuity and Poisson's equation for two oppositely charged streams of particles. Once again the frame of reference is chosen so that the wave pattern is stationary. We have:

the equations of motion
$$u_1 \frac{du_1}{dx} = \frac{\mu_1 Z_1 e}{m} E, \tag{5}$$

$$u_2 \frac{du_2}{dx} = \frac{\mu_2 Z_2 e}{m} E; \tag{6}$$

the equations of continuity
$$n_1 u_1 = \bar{n}_1 \bar{u}_1, \tag{7}$$

$$n_2 u_2 = \bar{n}_2 \bar{u}_2; \tag{8}$$

Poisson's equation
$$\frac{dE}{dx} = 4\pi e(n_1 Z_1 + n_2 Z_2), \tag{9}$$

and the condition of overall space-charge neutrality

$$\bar{n}_1 Z_1 + \bar{n}_2 Z_2 = 0. \tag{10}$$

* Langmuir (1929) has solved, by numerical integration, a problem of this kind for an ion-electron plasma.

The symbols u_r, μ_r, Z_r, n_r, x, $\bar{u}_r$ and $\bar{n}_r$ have the same meaning as they had in § 1, and further E = electric field intensity. The equations are made dimensionless by the substitutions

$$U_r = u_r/\bar{u}_r, \tag{11}$$

$$1+S_r = n_r/\bar{n}_r, \tag{12}$$

$$k^2 = \frac{4\pi e^2}{m}\left(\frac{\bar{n}_1\mu_1 Z_1^2}{\bar{u}_1^2}+\frac{\bar{n}_2\mu_2 Z_2^2}{\bar{u}_2^2}\right), \tag{13}$$

$$X = kx, \tag{14}$$

$$\tan^2\epsilon = \frac{\bar{n}_1\mu_1 Z_1^2}{\bar{u}_1^2}\bigg/\frac{\bar{n}_2\mu_2 Z_2^2}{\bar{u}_2^2}, \tag{15}$$

$$\mathscr{E} = \frac{k\sin\epsilon\,\cos\epsilon}{4\pi\bar{n}_1 Z_1 e}E; \tag{16}$$

in virtue of the condition of overall space charge neutrality (10), the relation (15) is equivalent to

$$\tan^2\epsilon = \frac{\mu_1|Z_1|}{\bar{u}_1^2}\bigg/\frac{\mu_2|Z_2|}{\bar{u}_2^2}. \tag{17}$$

Finally, one can assume without loss of generality that Z_1 is positive and that $\tan^2\epsilon \geqslant 1$, so that

$$\tfrac{1}{4}\pi \leqslant \epsilon \leqslant \tfrac{1}{2}\pi. \tag{18}$$

Equations (5) to (10) are then equivalent to

$$U_1\frac{dU_1}{dX} = \mathscr{E}\tan\epsilon, \tag{19}$$

$$U_2\frac{dU_2}{dX} = -\mathscr{E}\cot\epsilon, \tag{20}$$

$$U_1(1+S_1) = 1, \tag{21}$$

$$U_2(1+S_2) = 1, \tag{22}$$

and

$$\frac{d\mathscr{E}}{dX} = \left(\frac{1}{U_1}-\frac{1}{U_2}\right)\sin\epsilon\,\cos\epsilon. \tag{23}$$

It follows from (19) and (20) that

$$U_1^2\cot\epsilon + U_2^2\tan\epsilon = \text{constant} = A^2, \quad \text{say.} \tag{24}$$

Hence U_1 and U_2 can be expressed in the form

$$U_1 = A\cos\phi\tan^{\frac{1}{2}}\epsilon \tag{25}$$

and

$$U_2 = A\sin\phi\cot^{\frac{1}{2}}\epsilon. \tag{26}$$

We have restricted the discussion to the case in which no particles are trapped, so that all the particles in a given stream move in the same direction at all times. The dimensionless variables $U_r \equiv u_r/\bar{u}_r$ are therefore always positive, and it follows from relations (25) and (26) that $0 \leqslant \phi \leqslant \frac{1}{2}\pi$, if A is positive. With the aid of (25) and (26), equations (19) and (20) both reduce to

$$-A^2\sin\phi\cos\phi\frac{d\phi}{dX} = \mathscr{E}. \tag{27}$$

With the substitution
$$dX = \kappa \sin\phi \cos\phi \, d\tau \tag{28}$$
this equation becomes
$$-\frac{A^2}{\kappa}\frac{d\phi}{d\tau} = \mathscr{E}. \tag{29}$$

With the aid of (25), (26) and (28), equation (23) becomes

$$\begin{aligned}\frac{d\mathscr{E}}{d\tau} &= \frac{\kappa}{A}\sin\epsilon\cos\epsilon(\sin\phi\cot^{\frac{1}{2}}\epsilon - \cos\phi\tan^{\frac{1}{2}}\epsilon) \\ &= \frac{\kappa}{A}(\sin\epsilon\cos\epsilon)^{\frac{1}{2}}\sin(\phi-\epsilon).\end{aligned} \tag{30}$$

From (29) and (30) it follows that

$$\frac{A^3}{\kappa^2(\sin\epsilon\cos\epsilon)^{\frac{1}{2}}}\frac{d^2\phi}{d\tau^2} + \sin(\phi-\epsilon) = 0. \tag{31}$$

Let
$$\kappa = \frac{A^{\frac{3}{2}}}{(\sin\epsilon\cos\epsilon)^{\frac{1}{4}}}, \tag{32}$$
then
$$\frac{d^2\phi}{d\tau^2} + \sin(\phi-\epsilon) = 0. \tag{33}$$

This equation has the first integral

$$\frac{d\phi}{d\tau} = [2\{\cos(\phi-\epsilon) - \cos\alpha\}]^{\frac{1}{2}}, \tag{34}$$

where α is an arbitrary angle. Values of ϕ therefore lie in the range $(\epsilon-\alpha, \epsilon+\alpha)$. Since $\cos\phi$ must not change sign, the range of permissible values of α is limited by $\alpha+\epsilon \leqslant \frac{1}{2}\pi$, or
$$\alpha \leqslant \tfrac{1}{2}\pi - \epsilon. \tag{35}$$

But our variables were chosen so that $\frac{1}{4}\pi \leqslant \epsilon \leqslant \frac{1}{2}\pi$, and so

$$\alpha \leqslant \tfrac{1}{4}\pi. \tag{36}$$

The condition that $\sin\phi$ shall not change sign is given by $\epsilon-\alpha \geqslant 0$, and this always holds, since ϵ exceeds and α is less than $\frac{1}{4}\pi$.

Now $d\phi/d\tau$ vanishes at $\phi = \epsilon-\alpha$ and $\phi = \epsilon+\alpha$. The expression on the right-hand side of (34) can be positive or negative; it follows that ϕ is periodic in τ, and therefore also in X. In dimensionless units the wavelength of the oscillation described by these equations becomes

$$\begin{aligned}\Lambda &= 2\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha} dX = 2\kappa\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha}\sin\phi\cos\phi\,d\tau \\ &= 2A^{\frac{3}{2}}(\sin\epsilon\cos\epsilon)^{-\frac{1}{4}}\int_{\epsilon-\alpha}^{\epsilon+\alpha}\frac{\sin\phi\cos\phi\,d\phi}{[2\{\cos(\phi-\epsilon)-\cos\alpha\}]^{\frac{1}{2}}} \\ &= 2^{\frac{1}{4}}A^{\frac{3}{2}}\sin^{-\frac{1}{4}}2\epsilon\int_{-\alpha}^{\alpha}\frac{\sin 2(\psi+\epsilon)\,d\psi}{\{2(\cos\psi-\cos\alpha)\}^{\frac{1}{2}}} \\ &= 2^{\frac{3}{4}}\mathrm{A}^{\frac{3}{2}}\sin^{\frac{3}{4}}2\epsilon\int_0^{\alpha}\frac{\cos 2\psi\,d\psi}{(\cos\psi-\cos\alpha)^{\frac{1}{2}}} \\ &\equiv 2^{\frac{1}{4}}\pi A^{\frac{3}{2}}\sin^{\frac{3}{4}}2\epsilon\,I_2(\alpha).\end{aligned} \tag{37}$$

In (37), and from now on, we define

$$I_r(\alpha) = \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{\cos r\psi\, d\psi}{(\cos\psi - \cos\alpha)^{\frac{1}{2}}}. \tag{38}$$

A subsidiary condition to be satisfied is that the space average of S_1 shall vanish and therefore

$$\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha}\left(1-\frac{1}{U_1}\right)dX = 0. \tag{39}$$

This is equivalent to

$$\begin{aligned}\Lambda &\equiv 2\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha} dX = 2\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha}\frac{dX}{U_1}\\ &= 2A^{\frac{3}{2}}(\sin\epsilon\cos\epsilon)^{-\frac{1}{2}}\int_{\epsilon-\alpha}^{\epsilon+\alpha}\frac{\sin\phi\cos\phi\, d\phi}{A\cos\phi\tan^{\frac{1}{2}}\epsilon[2\{\cos(\phi-\epsilon)-\cos\alpha\}]^{\frac{1}{2}}}\\ &= 2^{\frac{3}{2}}A^{\frac{1}{2}}(\sin\epsilon\cos\epsilon)^{\frac{1}{2}}\int_0^\alpha\frac{\cos\psi\, d\psi}{(\cos\psi-\cos\alpha)^{\frac{1}{2}}}\\ &= 2^{\frac{1}{2}}\pi A^{\frac{1}{2}}\sin^{\frac{1}{2}}2\epsilon\, I_1(\alpha).\end{aligned} \tag{40}$$

Comparison of (37) and (40) now shows that the expression for A in terms of α is

$$A = 2^{\frac{1}{2}}\sin^{-\frac{1}{2}}2\epsilon\, I_1(\alpha)/I_2(\alpha). \tag{41}$$

Equation (37) then gives

$$\Lambda = 2\pi I_1^{\frac{3}{2}}(\alpha)/I_2^{\frac{1}{2}}(\alpha). \tag{42}$$

The total variation of electrostatic potential in the wave is, in dimensionless units,

$$\Delta\mathscr{V} = \left|\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha}\mathscr{E}\, dX\right|, \tag{43}$$

and by equation (29)

$$\begin{aligned}\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha}\mathscr{E}\, dX &= -A^2\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha} d\phi\frac{dX}{\kappa\, d\tau}\\ &= -A^2\int_{\epsilon-\alpha}^{\epsilon+\alpha}\sin\phi\cos\phi\, d\phi,\end{aligned} \tag{44}$$

by virtue of relation (28). Hence

$$\begin{aligned}\Delta\mathscr{V} &= \tfrac{1}{4}A^2\{\cos 2(\epsilon-\alpha)-\cos 2(\epsilon+\alpha)\}\\ &= \tfrac{1}{2}A^2\sin 2\epsilon\sin 2\alpha = \frac{I_1^2(\alpha)}{I_2^2(\alpha)}\sin 2\alpha,\end{aligned} \tag{45}$$

with the aid of (41).

Equations (42) and (45) express, in terms of the parameter α, the relation between the wavelength Λ and the maximum variation $\Delta\mathscr{V}$ of the potential.

Another quantity of interest is the space average $\langle U_r\rangle_{\text{sp}}$ of the particle velocities in the rth stream. This will be needed in later discussions of the stability of different kinds of plasma waves. For the particles in the first stream we have, in dimensionless units, that

$$\begin{aligned}\langle U_1\rangle_{\text{sp}} &= \frac{2}{\Lambda}\int_{\phi=\epsilon-\alpha}^{\phi=\epsilon+\alpha}U_1\, dX\\ &= \frac{2}{\Lambda}\int_{\epsilon-\alpha}^{\epsilon+\alpha}\frac{A\cos\phi\tan^{\frac{1}{2}}\epsilon\,\kappa\sin\phi\cos\phi\, d\phi}{[2\{\cos(\phi-\epsilon)-\cos\alpha\}]^{\frac{1}{2}}}\\ &= \frac{2^{\frac{1}{2}}A^{\frac{3}{2}}\sin^{\frac{1}{2}}\epsilon}{\Lambda\cos^{\frac{3}{2}}\epsilon}\int_{\epsilon-\alpha}^{\epsilon+\alpha}\frac{\sin\phi\cos^2\phi\, d\phi}{\{\cos(\phi-\epsilon)-\cos\alpha\}^{\frac{1}{2}}}.\end{aligned} \tag{46}$$

With the usual substitution $\phi = \psi + \epsilon$, with formulae (41) and (42) for Λ and A, and after some reductions, one finds that (46) becomes

$$\langle U_1 \rangle_{\rm sp} = \frac{I_1(\alpha)}{4I_2^2(\alpha)\sin\epsilon\cos^2\epsilon}\{I_1(\alpha)\sin\epsilon + I_3(\alpha)\sin 3\epsilon\}$$

$$= \frac{I_1(\alpha)}{I_2^2(\alpha)}[I_3(\alpha) + \tfrac{1}{4}\{I_1(\alpha) - I_3(\alpha)\}\sec^2\epsilon]; \tag{47}$$

similarly

$$\langle U_2 \rangle_{\rm sp} = \frac{I_1(\alpha)}{I_2^2(\alpha)}[I_3(\alpha) + \tfrac{1}{4}\{I_1(\alpha) - I_3(\alpha)\}\operatorname{cosec}^2\epsilon]. \tag{48}$$

3. Discussion of formulae and some numerical results

We first consider the results predicted in the limit $\Delta\mathscr{V} \to 0$. This has two possible physical meanings, for it follows from formulae (14) and (16) that, in terms of the physical variables,

$$\Delta\mathscr{V} = \frac{k^2 \sin\epsilon\,\cos\epsilon\,\Delta V}{4\pi\bar{n}_1 Z_1 e}. \tag{49}$$

Therefore the limit may imply that $\Delta V \to 0$, so that the variation of the physical potential is small, and the properties of the wave can be discussed by means of the linearized equations. Alternatively the limit may imply that $\cos\epsilon \to 0$, so that $\tan\epsilon \to \infty$, and by relation (17), that

$$\mu_2|Z_2|/\bar{u}_2^2 \to 0. \tag{50}$$

(The possibility that $\sin\epsilon$ tends to zero is excluded, because ϵ must exceed $\frac{1}{4}\pi$.) The physical meaning of (50) is that the particles in the second stream either have a large mass m/μ_2, or a small specific charge $Z_2 e$, or a high mean velocity $\bar{u}_2$ relative to the wave pattern. In any of these cases their motion is only slightly disturbed by the electrostatic field of the plasma wave; the second stream therefore essentially acts as a neutralizing background charge to an oscillation, possibly one of finite amplitude, which is taking place in the first stream.

It is readily shown that $\Delta\mathscr{V}$ and α tend to zero together, since, for small α,

$$I_r(\alpha) \doteqdot \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{d\psi}{\{\frac{1}{2}(\alpha^2 - \psi^2)\}^{\frac{1}{2}}} = 1, \tag{51}$$

for all values of r, and it follows from (45) that then

$$\Delta\mathscr{V} \doteqdot \sin 2\alpha \doteqdot 2\alpha. \tag{52}$$

Relations (42) and (51) show that, in this limit

$$\Lambda = 2\pi. \tag{53}$$

The wavelength of such oscillations, expressed in physical units, is

$$\lambda = \frac{2\pi}{k}, \tag{54}$$

with k given by (13). Our result therefore agrees with that of linearized theory in the case $\Delta V \to 0$. Our prediction, in the case $\mu_2|Z_2|/\bar{u}_2^2 \to 0$, is the same as in (54), with k now given by

$$k^2 = \frac{4\pi\bar{n}_1 e^2 \mu_1 Z_1^2}{m\bar{u}_1^2}. \tag{55}$$

Here the wavelength of the oscillation is independent of the amplitude, given the nature and density of the particles in the first stream, and their mean speed relative to the wave pattern. This prediction agrees with the results found by Polovin (1956) in his note on exact non-linear plasma oscillations of a single stream of particles—Polovin actually shows that, viewed from a frame of reference in which the plasma has no mean velocity, the frequency of the oscillation is independent of its amplitude. Now in dimensionless units this frequency is $2\pi/\Lambda$; our result is therefore equivalent to Polovin's.

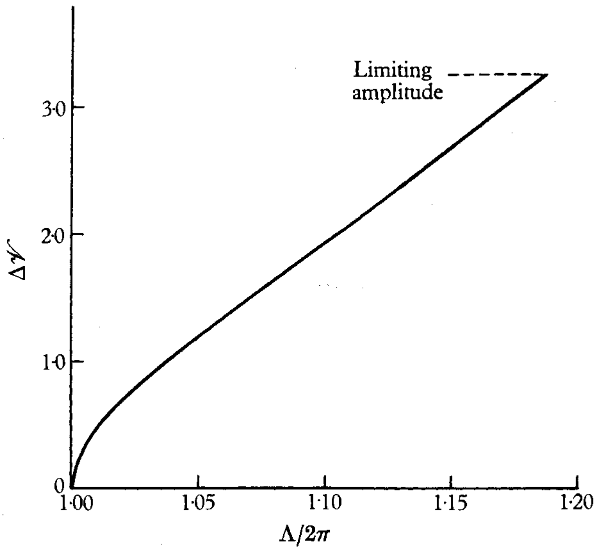

FIGURE 1. A plot of $\Delta\mathscr{V}$ against $\Lambda/2\pi$.

In a general case, with given values for μ_r, Z_r, $\bar{n}_r$ and $\bar{u}_r$, the wavelength is an increasing function of α, and therefore of the potential variation $\Delta\mathscr{V}$. The relation between $\Lambda/2\pi$ and $\Delta\mathscr{V}$ is readily found from equations (42) and (45); the functions $I_1(\alpha)$ and $I_2(\alpha)$ are expressible in terms of complete elliptic integrals of the first and second kind (see §290, Byrd & Friedman 1954). In the present case, with α restricted to the range $(0, \frac{1}{4}\pi)$, the I_r functions may also be expressed, to better than one per cent accuracy, in terms of elementary functions, and this is done in the appendix.

The table below gives some representative values of $\Lambda/2\pi$ in terms of $\Delta\mathscr{V}$, and figure 1 shows graphically how the two quantities are related. It will be seen that, for values of $\Delta\mathscr{V}$ larger than about 0·7, the dispersion relation is very well approximated by

$$\frac{\Lambda}{2\pi} = 1 + 0{\cdot}0654(\Delta\mathscr{V} - 0{\cdot}400). \tag{56}$$

For smaller values of $\Delta\mathscr{V}$, and therefore of α, one can see, from the formulae in the appendix, that

$$I_1(\alpha) \doteqdot 1 - 3\alpha^2/16 \quad \text{and} \quad I_2(\alpha) = 1 - 15\alpha^2/16, \tag{57}$$

so that

$$\frac{\Lambda}{2\pi} \doteqdot 1 + \frac{3\alpha^2}{16}. \tag{58}$$

In this range

$$\Delta\mathscr{V} \equiv \frac{I_1^2(\alpha)}{I_2^2(\alpha)}\sin 2\alpha \doteqdot \left(1+\frac{3\alpha^2}{2}\right)\left(2\alpha-\frac{4\alpha^3}{3}\right),$$

and so

$$\alpha \doteqdot \frac{\Delta\mathscr{V}}{2}\left\{1-\frac{5(\Delta\mathscr{V})^2}{24}\right\}. \tag{59}$$

It follows from (58) and (59) that the approximate dispersion relation for small $\Delta\mathscr{V}$, is

$$\frac{\Lambda}{2\pi} \doteqdot 1+\frac{3(\Delta\mathscr{V})^2}{64}\left\{1-\frac{5(\Delta\mathscr{V})^2}{12}\right\}. \tag{60}$$

A table of selected values of the dimensionless wavelength $\Lambda/2\pi$ in terms of the dimensionless potential variation $\Delta\mathscr{V}$. The maximum possible value of $\Delta\mathscr{V}$ in a double stream is 3·284 *if there are to be no trapped particles.*

$\Delta\mathscr{V}$	0·694	1·089	1·498	1·983	2·589	3·284
$\Lambda/2\pi$	1·020	1·043	1·073	1·104	1·144	1·188

Appendix: Elementary expressions for $I_r(\alpha)$

Formulae for $I_r(\alpha)$, correct to better than 1 % in the range $0 \leqslant \alpha \leqslant \frac{1}{4}\pi$, can be worked out as follows.

We have

$$I_r(\alpha) = \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{\cos r\psi\, d\psi}{(\cos\psi-\cos\alpha)^{\frac{1}{2}}}. \tag{61}$$

For integral values of r, $\cos r\psi$ can be expressed as a polynomial in terms of $\cos\psi$, say $\cos r\psi = F_r(\cos\psi)$. Let

$$\cos\psi = 1-2\sin^2\tfrac{1}{2}\alpha\sin^2\theta, \tag{62}$$

then $\theta = 0$ when $\psi = 0$, and $\theta = \frac{1}{2}\pi$ when $\psi = \alpha$. Further

$$\cos\psi-\cos\alpha = 2\sin^2\tfrac{1}{2}\alpha\cos^2\theta, \tag{63}$$

$$\sin\psi = 2\sin\tfrac{1}{2}\alpha\sin\theta(1-\sin^2\tfrac{1}{2}\alpha\sin^2\theta)^{\frac{1}{2}}, \tag{64}$$

and

$$d\psi = \frac{2\sin\frac{1}{2}\alpha\cos\theta\, d\theta}{(1-\sin^2\frac{1}{2}\alpha\sin^2\theta)^{\frac{1}{2}}}. \tag{65}$$

Thus

$$I_r(\alpha) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{F_r(1-2\sin^2\frac{1}{2}\alpha\sin^2\theta)}{(1-\sin^2\frac{1}{2}\alpha\sin^2\theta)^{\frac{1}{2}}}\,d\theta. \tag{66}$$

In the present application $\alpha \leqslant \frac{1}{4}\pi$, and therefore

$$\sin^2\tfrac{1}{2}\alpha\sin^2\theta \leqslant \tfrac{1}{4}(2-\sqrt{2}) \doteqdot 0{\cdot}147,$$

for all possible θ. In writing

$$(1-\sin^2\tfrac{1}{2}\alpha\sin^2\theta)^{-\frac{1}{2}} \doteqdot 1+\tfrac{1}{2}\sin^2\tfrac{1}{2}\alpha\sin^2\theta \tag{67}$$

we thus make an error smaller than

$$\tfrac{3}{8}\sin^4\tfrac{1}{2}\alpha \leqslant 0{\cdot}0081. \tag{68}$$

Thus the integrand in (66) can be well approximated by a polynomial in $\sin^2\theta$; when this is done, the integration is straightforward.

After some reductions the following expressions are found for $I_0(\alpha)$, $I_1(\alpha)$, $I_2(\alpha)$ and $I_3(\alpha)$:

$$I_0(\alpha) \doteqdot (9-\cos\alpha)/8, \tag{69}$$

$$I_1(\alpha) \doteqdot (17+18\cos\alpha-3\cos^2\alpha)/32, \tag{70}$$

$$I_2(\alpha) \doteqdot (-11+21\cos\alpha+27\cos^2\alpha-5\cos^3\alpha)/32, \tag{71}$$

$$I_3(\alpha) \doteqdot (-39-116\cos\alpha+138\cos^2\alpha+180\cos^3\alpha-35\cos^4\alpha)/128. \tag{72}$$

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